EXTENSIONS OF ÉTALE BY CONNECTED GROUP SPACES

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ABSTRACT. The main theorem, in rough terms, asserts the following. Let K and D be group spaces over a scheme S. Assume that K has connected fibers and that D is finite and étale over S. Assume that there exists a single finite, surjective, étale, Galois morphism $\overline{S} \to S$ which decomposes (scheme-theoretically) every extension of D by K. Let $\pi = \mathrm{Aut}(\overline{S}/S)$. Then group space extensions of D with kernel K are in bijective correspondence with pairs (ξ, χ) consisting of a π-group extension

$$\xi\colon 1 \to K(\overline{S}) \to X \to D(\overline{S}) \to 1$$

and a π-group homomorphism $\chi\colon X \to \mathrm{Aut}(\overline{K})$ which lifts the conjugation map $X \to \mathrm{Aut}(K(\overline{S}))$ and which agrees with the conjugation map $K(\overline{S}) \to \mathrm{Aut}(\overline{K})$. In this way, the calculation of group space extensions is reduced to a purely group-theoretic calculation.

INTRODUCTION

Fix a connected noetherian scheme S, and consider group spaces over S, i.e. group objects in the category of algebraic spaces over S. We shall assume without further comment that all group spaces under consideration are flat and locally of finite type over S. Consider the problem of classifying all group spaces G over S. Stated in such generality, this problem is absurdly difficult. Nevertheless, it seems worthwhile to consider certain aspects of this problem in full generality. In this paper we consider a key ingredient of the classification problem, which we refer to as the *connectedness problem*: for a given group space G, how do the "connected pieces" of G fit together to form G itself? We will make this question precise. Roughly, the problem is to classify all group spaces, assuming that one already knows how to classify all group spaces having geometrically connected fibers.

The easiest situation to describe is that in which G is *decomposable*, in the sense that it may be expressed as a disjoint union of copies of a fixed algebraic space having geometrically connected fibers over S. Obviously, if G has geometrically connected fibers, then it is decomposable.

To make any progress, we shall require that G is *locally decomposable*, i.e. that there exists a faithfully flat quasicompact (ffqc) morphism $\overline{S} \to S$ which *decomposes* G in the sense that $\overline{G} = G \times_S \overline{S}$ is decomposable over $\overline{S}$.

Although not every group space is locally decomposable, the restriction imposed thusly on G seems natural. This restriction is equivalent to requiring that there exist an extension of group spaces:

$$1 \to K \to G \to D \to 1$$

in which K has geometrically connected fibers, D is étale, and D is itself locally decomposable. We call such a D an *étale covering group*. It is the algebraic analog of a covering space.

Thus we may give a precise formulation to the connectedness problem: classify étale covering groups D, and determine all extensions of D by K, where K has geometrically connected fibers.

In §1 we show that when S is geometrically unibranch, the problem of classifying étale covering groups has a nice, easy answer: the category of such objects is equivalent to the category of $\pi_1(S)$-groups which have finite orbits. Forgetting about the group structure, such an object is just a disjoint union of finite étale covers. As for the extension problem, the main result of this paper asserts that under suitable hypotheses, this problem admits a purely group-theoretic formulation. The details are given later in the introduction.

To make any further progress, we need to assume that G is locally decomposable in the strong sense that there exists a finite surjective étale morphism $\overline{S} \to S$ which decomposes G. If G has this property, we shall say that it is *isodecomposable*.

It follows that if S is geometrically unibranch, G is an étale covering group, and the fibers of G are finitely generated, then G is isodecomposable.

From now on we fix K (having geometrically connected fibers), and D, an isodecomposable étale covering group. We consider the extension problem.

For suitable D and K, it may be true that every extension of D by K is isodecomposable. This is true under the following conditions: (I) S is the spectrum of a perfect field; or (II) S is the spectrum of an Artin local ring and K is smooth; or [considering only commutative extensions] (III) D is finite, $|D|$ is relatively prime to the residue characteristics of S, K is separated, smooth, commutative, and nondegenerate, in the sense that the abelian, multiplicative, and unipotent ranks of the fibers of K are constant as functions on S.

It is often possible to find a single finite surjective étale morphism $\overline{S} \to S$ which decomposes every extension of D by K. This is not always true: there exist counterexamples in case (I) where S is the spectrum of a number field, K is an elliptic curve, and $D = \mathbb{Z}/2\mathbb{Z}$.

Assuming the existence of such an $\overline{S}$, which we may assume to be Galois, we show that the computation of

$$\{\text{group space extensions of } D \text{ with kernel } K\}$$

may be reduced to a purely group theoretic calculation. More generally, if we assume only that $\overline{S}$ decomposes D, then the computation of

$$\{\overline{S}\text{-decomposed group space extensions of } D \text{ with kernel } K\}$$

may be reduced to the same sort of calculation. Let $\pi = \text{Aut}(\overline{S}/S)$. We show that such group space extensions are in bijective correspondence with pairs (ξ, χ) consisting of a π-group extension

$$\xi: 1 \to K(\overline{S}) \to X \to D(\overline{S}) \to 1$$

and a π-group homomorphism $\chi \colon X \to \mathrm{Aut}(\overline{K})$ which lifts the conjugation map $X \to \mathrm{Aut}(K(\overline{S}))$ and which agrees with the conjugation map $K(\overline{S}) \to \mathrm{Aut}(\overline{K})$. Of course, the precise statement must refer to isomorphism classes. Also, we shall need to assume that S is of finite type over some excellent Dedekind domain, because this hypothesis is used in a representability theorem of Artin which we refer to.

This group-theoretic interpretation of group space extensions was known in the special case where E is commutative, D is decomposable, and only decomposable extensions are considered; see e.g. [5, III, §6, 4.2]. Other references on group scheme extensions include [15, 18, 20, 21, and 22].

To help put all of this in perspective, we recall in §7 the various (known) obstructions which prevent a group space from being isodecomposable.

Conventions. The symbol S will always denote a connected noetherian scheme, of finite type over some excellent Dedekind domain, or affine and essentially of finite type over some excellent Dedekind domain. Let X be an algebraic space over S. If $f \colon S' \to S$ is a ffqc morphism, we shall say that f *decomposes* X if $X' = X \times_S S'$ may be expressed as a disjoint union of algebraic spaces, all isomorphic (as S'-spaces) and having geometrically connected fibers over S'. If such an S' exists, we say that X is *locally decomposable*. In particular we apply this terminology to group spaces X. If one may take S' to be finite étale surjective, then we say that X is *isodecomposable*. If G/S is a group space, and G acts on an algebraic space X/S, and $G' \cong X'$ via this action for some fppf morphism $S' \to S$, we say that X is a *principal homogeneous space* under G. We let $\mathrm{PHS}(G)$ denote the set of isomorphism classes of principal homogeneous spaces under G. If G is commutative, $\mathrm{PHS}(G)$ may be given the structure of an abelian group.

1. ÉTALE COVERING GROUPS

An *étale cover* X of S is an algebraic space X over S which is étale and locally decomposable. Such an X is necessarily a scheme, e.g. by [12, II, 6.16]. For example, any finite étale morphism is an étale covering space. An *étale covering group* is an étale cover which is also a group space.

Proposition 1.1. *Assume that S is geometrically unibranch. Let $\bar{s}$ be a geometric point of S. Then the category of étale covering spaces of S is equivalent to the category of $\pi_1(S, \bar{s})$-sets X which have finite orbits.*

Remark 1.2. The action of $\pi_1(S, \bar{s})$ on X is assumed to be continuous with respect to the discrete topology on X and the profinite topology on $\pi_1(S, \bar{s})$. This condition is often automatically satisfied, for instance if $\pi_1(S, \bar{s})$ is the profinite completion of some group.

Proof. Let $\pi \colon Y \to S$ be an étale covering space. First assume that Y is connected. Since S is geometrically unibranch, one knows by [10, 18.10.1] that Y is locally integral, and hence irreducible. We will show that π is finite. Choose an open quasicompact subset $Y_0 \subset Y$ such that Y_0 surjects onto S. There exists an open dense subset $U \subset S$ such that $Y_0 \to S$ is finite over U. It follows by [8, 6.1.5(v)] that $Y_0 \to Y$ is finite over U, and hence that $Y_0 \cap \pi^{-1}(U)$ is a closed (and open) subset of $\pi^{-1}(U)$.

Since Y is irreducible, $\pi^{-1}(U)$ is connected. Hence $Y_0 \cap \pi^{-1}(U) = \pi^{-1}(U)$. Hence $\pi^{-1}(U)$ is quasicompact. Hence after a ffqc base change $\overline{S} \to S$, $\overline{Y}$ is a disjoint union of finitely many copies of $\overline{S}$. By faithfully flat descent [9, 2.7.1], π is finite.

Now let $\pi: Y \to S$ be an arbitrary étale covering space. By the above calculation, we see that π is a disjoint union of finite étale covering spaces. For each member of this union, we obtain a $\pi_1(S, \overline{s})$-set in the usual way. The union of these sets is a $\pi_1(S, \overline{s})$-set which has finite orbits. Conversely, given a $\pi_1(S, \overline{s})$-set with finite orbits, one obtains an étale covering space in the obvious way. $\square$

Example 1.3. If S is not geometrically unibranch, it need not be true that the category of étale covering spaces of S is equivalent to the category of $\pi_1(S, \overline{s})$-sets X which have finite orbits. (This observation seems implicit in the proof of [19, XIII, 3.1].) Indeed, let $S = (\mathrm{Spec}\ \mathbb{C}[x, y, z]/(xy, z)) - \{(0, \pm 1, 0), (\pm 1, 0, 0)\}$, thought of as a closed subscheme of $\mathbb{A}_{\mathbb{C}}^3$. We construct an étale covering space $X \to S$. For each $n \in \mathbb{Z}$, let C_n be the locally closed subscheme of $\mathbb{A}_{\mathbb{C}}^3$ given by $y = 0$ and $x^2 + (z - 4n)^2 = 1$, less the points $(\pm 1, 0, 4n)$. Let D_n be the locally closed subscheme of $\mathbb{A}_{\mathbb{C}}^3$ given by $x = 0$ and $y^2 + (z - 2 - 4n)^2 = 1$, less the points $(0, \pm 1, 4n + 2)$. Each C_n and each D_n maps via projection to S. Let $\widetilde{X}$ be the disjoint union of all the C_n and the D_n. Then we have an induced morphism $\widetilde{X} \to S$. Define a scheme X by taking $\widetilde{X}$ and glueing C_n to D_n (transversally) at the point $(0, 0, 4n + 1)$, for each n. As C_n in fact meets D_n transversally at this point in $\mathbb{A}_{\mathbb{C}}^3$, we have an induced morphism $\pi: X \to S$. Note that e.g. C_n meets C_{n-1} with respect to their embeddings in $\mathbb{A}_{\mathbb{C}}^3$, but that the images of C_n and C_{n-1} in X do not meet.

It is pleasing to consider the set $X(\mathbb{R})$, which may be thought of as a *lamp chain*, with successive links oriented in alternating directions, extending infinitely in both directions. (Actually, since the points $(0, \pm 1, *)$ and $(\pm 1, 0, *)$ are missing, the chain will fall apart in the usual topology.) One may view $X(\mathbb{R})$ as sitting in $\mathbb{R}^3$, but of course X is not a closed subscheme of $\mathbb{A}_{\mathbb{C}}^3$.

We will show that $\pi: X \to S$ is an étale covering space. Since X is connected and not quasi-compact, it will be a counterexample to the stated equivalence of categories. It suffices to show that π is an analytic covering space. This may be done locally in the usual topology on S. There is no difficulty away from $(0, 0, 0) \in S$. Over a small neighborhood U of $(0, 0, 0)$, X decomposes as an infinite disjoint union of pieces, each isomorphic to U. The reader who finds this specious may prefer to argue that X decomposes as an infinite disjoint union of pieces, each mapping finitely to U. Then, by explicit calculation one may verify that the said mappings are étale. In any event, one finds that π is an étale covering space.

Corollary 1.4. *Let S be a connected noetherian scheme. Assume that S is geometrically unibranch. Let $\overline{s}$ be a geometric point of S. Then the category of étale covering groups of S is equivalent to the category of $\pi_1(S, \overline{s})$-groups X which have finite orbits.*

Remark 1.5. This is probably not true if S is not geometrically unibranch, but we do not know of a counterexample.

Corollary 1.6. *Let S be a connected noetherian scheme. Assume that S is geometrically unibranch. Let G be an étale covering group over S whose geometric fibers are finitely generated. Then G is isodecomposable.*

Proof. Let G correspond to the $\pi_1(S, \bar{s})$-group H. Let $h_1, \ldots, h_n$ be generators for H as a group. Since the orbits of H under $\pi_1(S, \bar{s})$ are finite, it follows that $\mathrm{Fix}(h_i)$ is a closed subgroup of finite index for each i. Hence $\mathrm{Fix}(h_1) \cap \cdots \cap \mathrm{Fix}(h_n)$ is a closed subgroup of finite index. Hence the action of $\pi_1(S, \bar{s})$ on H factors through a finite quotient of $\pi_1(S, \bar{s})$. Hence base extension by the corresponding Galois extension of S will decompose G. $\square$

Example 1.7. An étale covering group G whose geometric fibers are not finitely generated need not be isodecomposable. For an example, choose any sequence $G_1, G_2, \ldots$ of finite abelian étale covering groups of S which are not decomposed by any finite étale extension. Let $G = \bigoplus_{i=1}^{\infty} G_i$. If H_i and H are the corresponding $\pi_1(S, \bar{s})$-groups, then $H = \bigoplus_{i=1}^{\infty} H_i$. Then H has finite orbits, so G is an étale covering space, but it is not isodecomposable.

2. CONNECTIONS WITH PRINCIPAL HOMOGENEOUS SPACES

In this section we explore some connections between the extension problem and principal homogeneous spaces. This will help us to construct examples of extensions.

Given an extension of group spaces

$$1 \to K \to G \to D \to 1$$

where D is discrete, we obtain a bunch of principal homogeneous spaces under K by taking the cosets of K in G.

When K is commutative, we can reverse this construction, in the following sense. For any fppf morphism $\mathscr{U} \to S$, there are groups $C^0(\mathscr{U}, K) = K(\mathscr{U})$, $C^1(\mathscr{U}, K) = K(\mathscr{U} \times \mathscr{U})$, and so forth which in fact may be assembled into a complex $C^{\bullet}(\mathscr{U}, K)$, the Čech complex of K relative to $\mathscr{U}$. Taking the direct limit as $\mathscr{U}$ varies, we obtain a complex $C^{\bullet}(S, K)$ whose first cohomology is $H^1(S_{\mathrm{fppf}}, K)$, the cohomology of K relative to the small fppf site. (According to [16, III, 3.9, 3.11(b)], one gets the same result if one computes cohomology relative to the big fppf site.) Using [16, III, 4.6] and Artin's representability theorems [1, 7.1, 7.2]; [2, 6.3], one concludes that $H^1(S_{\mathrm{fppf}}, K)$ may be identified with the set of isomorphism classes of principal homogeneous spaces under K. Let $Z^i(S, K)$ denote the kernel of the coboundary map $C^i(S, K) \to C^{i+1}(S, K)$.

Proposition 2.1. *Let K be a commutative group space over S. Give $Z^1(S, K)$ the discrete group space structure. Then there exists an extension*:

$$0 \to K \to G \to Z^1(S, K) \to 0$$

of commutative group spaces over S such that the coset of K in G coming from $x \in Z^1(S, K)$ is isomorphic to the corresponding principal homogeneous space.

Sketch. The Čech complex $C^{\bullet}(S, K)$ extends to a complex $\underline{C}^{\bullet}(S, K)$ of sheaves on the small fppf site over S. We obtain an exact sequence of sheaves on this site:

$$0 \to \underline{K} \to \underline{C}^0(S, K) \xrightarrow{\pi} \underline{Z}^1(S, K) \to 0.$$

Let $\underline{Z}_g^1(S, K)$ denote the subsheaf of $\underline{Z}^1(S, K)$ generated by its global sections. Let $\mathscr{F} = \pi^{-1}(\underline{Z}_g^1(S, K))$. We have an exact sequence

$$0 \to \underline{K} \to \mathscr{F} \to \underline{Z}_g^1(S, K) \to 0.$$

Clearly $\underline{Z}_g^1(S, K)$ is the constant sheaf associated to $Z^1(S, K)$. The sheaf $\mathscr{F}$ is representable by an algebraic space G. For further explanations, see the proof of [16, III, 4.6]. □

Obviously, given any subgroup $W \subset Z^1(S, K)$, one can use this to construct an extension of W by K. This would prove (2.2) (below), under the additional hypothesis that $o(X) = \infty$ in $\mathrm{PHS}(K)$. However, we can do better, without using (2.1). Indeed, the set $\mathrm{PHS}(K)$ may be identified with $H^1(S_{\mathrm{fppf}}, K)$, which may further be identified with $\mathrm{Ext}^1(\mathbb{Z}_S, K)$, where $\mathbb{Z}_S$ denotes the constant (fppf) sheaf on S corresponding to $\mathbb{Z}$, and Ext^1 denotes extensions of fppf sheaves. By Artin's representability theorems, we know that all such extensions are representable by algebraic spaces. Hence we have

Proposition 2.2. *Let K be a commutative group space over S. Let X be a principal homogeneous space under K. Then there exists an extension of commutative group spaces over S:*

$$0 \to K \to G \xrightarrow{\pi} \mathbb{Z} \to 0$$

such that $[\pi^{-1}(1)] = [X]$ in $\mathrm{PHS}(K)$.

3. GROUP TERMINOLOGY

Let π be a group. By a π-*group*, we mean a group G, together with a group homomorphism $\pi \to \mathrm{Aut}(G)$. A *morphism* of π-groups is a group homomorphism which commutes with the action of π.

If G is a π-group, then the group $\mathrm{Aut}(G)$ becomes a π-group via conjugation: $(x\sigma)(g) = x(\sigma(x^{-1}(g)))$, where $x \in \pi$, $\sigma \in \mathrm{Aut}(G)$, and $g \in G$. We shall always give $\mathrm{Aut}(G)$ this canonical π-group structure.

If K and D are π-groups, an *extension of D with kernel K* is an exact sequence

$$\xi: 1 \to K \to G \to D \to 1$$

of π-groups.

Given such an extension ξ, there is an induced π-homomorphism $\chi(\xi): G \to \mathrm{Aut}(K)$, induced by conjugation: $\chi(\xi)(g)(k) = gkg^{-1}$, where $g \in G$ and $k \in K$.

Let K, D, and H be π-groups. Let $\phi: H \to \mathrm{Aut}(K)$ and $\psi: K \to H$ be π-homomorphisms. In the situations we are interested in, $\phi \circ \psi$ will be the conjugation map, but it is not necessary to impose this as an axiom. An *H-restricted extension of D with kernel K* is a triple (G, ξ, χ) consisting of a π-group G, a π-group extension

$$\xi: 1 \to K \xrightarrow{i} G \to D \to 1,$$

and a π-homomorphism $\chi: G \to H$, such that $\chi(\xi) = \phi \circ \chi$ and $\psi = \chi \circ i$. (It would be more precise to refer to (H, ϕ, ψ)-restricted extensions.)

Given another such extension (G', ξ', χ'), we say that (G, ξ, χ) is *equivalent* to (G', ξ', χ') if there exists a π-isomorphism $\alpha \colon G \to G'$ such that

$$
\begin{array}{ccccccccc}
\xi : & 1 & \to & K & \to & G & \to & D & \to & 1 \\
 & & & \downarrow= & & \downarrow\alpha & & \downarrow= & & \\
\xi' : & 1 & \to & K & \to & G' & \to & D & \to & 1
\end{array}
$$

commutes and $\chi = \chi' \circ \alpha$.

4. EXTENSIONS OF ÉTALE COVERING GROUPS

Let K, D be group spaces over S. Assume that K has geometrically connected fibers. Assume that D is an isodecomposable étale covering group. Let $\overline{S} \to S$ be a finite étale cover which is Galois and which decomposes D. Let $\pi = \mathrm{Aut}(\overline{S}/S)$. Then $K(\overline{S})$ and $D(\overline{S})$ are π-groups. Let $\overline{K} = K \times_S \overline{S}$. Let $\mathrm{Aut}(\overline{K})$ denote the group of group space automorphisms of $\overline{K}/\overline{S}$. Given any $\beta \in \pi$, there is a canonical morphism of algebraic spaces $\tilde{\beta} \colon \overline{K} \to \overline{K}$ such that the diagram

$$
\begin{array}{ccc}
\overline{K} & \xrightarrow{\;\mathrm{can}\;} & \overline{S} \\
\Big\downarrow{\scriptstyle\tilde{\beta}} & & \Big\downarrow{\scriptstyle\beta} \\
\overline{K} & \xrightarrow{\;\mathrm{can}\;} & \overline{S}
\end{array}
$$

is Cartesian. Note that $\tilde{\beta}$ is not an $\overline{S}$-morphism. Nevertheless, if $\rho \in \mathrm{Aut}(\overline{K})$, then $\tilde{\beta} \circ \rho \circ \tilde{\beta}^{-1} \in \mathrm{Aut}(\overline{K})$. In this way $\mathrm{Aut}(\overline{K})$ becomes a π-group, and there is a canonical π-homomorphism $\phi \colon \mathrm{Aut}(\overline{K}) \to \mathrm{Aut}(K(\overline{S}))$. There is also a canonical π-homomorphism $\psi \colon K(\overline{S}) \to \mathrm{Aut}(\overline{K})$ given by conjugation.

Taking account of the above explanations we may write

Theorem 4.1. *Let K, D be group spaces over S. Assume that K has geometrically connected fibers. Assume that D is an isodecomposable étale covering group. Let $\overline{S} \to S$ be a finite étale cover which is Galois and which decomposes D. Let $\pi = \mathrm{Aut}(\overline{S}/S)$. Then there is a canonical bijective map from the set of equivalence classes of $\overline{S}$-decomposed group space extensions of D with kernel K to the set of equivalence classes of $\mathrm{Aut}(\overline{K})$-restricted π-group extensions of $D(\overline{S})$ with kernel $K(\overline{S})$.*

Remark 4.2. The statement is false if one replaces (everywhere) the word "group space" by "group scheme.". Indeed, Raynaud [19, XIII, 3.2] has constructed an example of the following: a noetherian normal local scheme S of dimension two, an abelian scheme A/S, and a principal homogeneous space X under A which is isotrivial but not a scheme. Using this and (2.2), one may construct an extension

$$
0 \to A \to G \to \mathbb{Z} \to 0
$$

of commutative group spaces in which G is not a scheme.

Proof. It is easy to construct the map of the theorem. The hard part is to prove that it is bijective, which we do by explicitly constructing its inverse. Let

$$
\xi \colon 1 \to K \to G \to D \to 1
$$

be an $\overline{S}$-decomposed (group space) extension of D with kernel K. Let $\overline{G} = G \times_S \overline{S}$ and $\overline{D} = D \times_S \overline{S}$. Then we have an exact sequence

$$1 \to \overline{K} \to \overline{G} \xrightarrow{p} \overline{D} \to 1$$

of group spaces over $\overline{S}$. By assumption $\overline{D}$ is decomposable. Also by assumption, the fibers of p are all isomorphic. This implies that $p(\overline{S})$ is surjective. As $\overline{D}(\overline{S}) = D(\overline{S})$ (and so forth), we obtain an extension of groups

$$\eta: 1 \to K(\overline{S}) \to G(\overline{S}) \xrightarrow{f} D(\overline{S}) \to 1.$$

It is clear that in fact this is an extension of π-groups. For any $g \in G(\overline{S}) = \overline{G}(\overline{S})$, we obtain by conjugation a π-morphism $G(\overline{S}) \to \mathrm{Aut}(\overline{K})$. This defines an $\mathrm{Aut}(\overline{K})$-restricted extension of $D(\overline{S})$ with kernel $K(\overline{S})$. Moreover, the equivalence class of this $\mathrm{Aut}(\overline{K})$-restricted extension does not change if we replace ξ by an equivalent extension.

Thus we have defined the map of the theorem. To prove that it is bijective, we define its inverse. Thus suppose given an extension of π-groups

$$\eta: 1 \to K(\overline{S}) \to X \xrightarrow{g} D(\overline{S}) \to 1$$

and a π-homomorphism $\sigma: X \to \mathrm{Aut}(\overline{K})$ with the desired property. (For typographical reasons we use the symbol σ instead of χ.) We let the group operation on X be denoted by $\star$. If $x \in X$, then x^{-1} denotes the inverse of x relative to $\star$. (We mention this explicitly because in the course of the proof, elements of X will enter into three separate operations.)

For each $d \in D(\overline{S})$, choose some $x_d \in g^{-1}(d)$. We may assume that $x_1 = 1$. Unfortunately, because of the possibility that $d^2 = 1$, we may not assume that $x_{d^{-1}} = (x_d)^{-1}$. This will make our calculations significantly harder. We may however assume at least that x_d commutes with $x_{d^{-1}}$. Define an $\overline{S}$-space:

$$H = \bigcup_{d \in D(\overline{S})} \overline{K}_d$$

where $\overline{K}_d$ is a copy of $\overline{K}$.

In the following, T will denote an algebraic space over $\overline{S}$.

If $x \in X$, then σ_x denotes the corresponding element of $\mathrm{Aut}(\overline{K})$. Actually, we can make σ_x act on H, as follows: if $y \in H(T)$, say $y \in \overline{K}_d(T)$, then the notation $\sigma_x(y)$ means: treat y as an element of $\overline{K}(T)$, then apply σ_x, then think of the result as an element of $\overline{K}_d(T)$.

We will define a group structure on H, by making $H(T)$ into a group with operation "$*$".

First we define an associative binary operation on $H(T)$, denoted by juxtaposition, and defined as follows. Let $y_1, y_2 \in H(T)$. We may write $y_1 \in \overline{K}_{d_1}(T)$, $y_2 \in \overline{K}_{d_2}(T)$ for suitable $d_1, d_2 \in D(\overline{S})$. Then $y_1 y_2$ is computed by thinking of y_1, y_2 as elements of the group $\overline{K}(T)$, multiplying them there, and then treating the result $y_1 y_2$ as an element of $\overline{K}_{d_1 d_2}(T)$.

Now we define a map of sets $\lambda: X \to H(\overline{S})$. Indeed if $w \in g^{-1}(d)$, we define $\lambda(w) = [w \star x_d^{-1}]_d$, where the notation $[\quad]_d$ means "treat the given element of $\overline{K}(\overline{S})$ as an element of $\overline{K}_d(\overline{S})$." Via λ, we identify elements $x \in X$ with

the corresponding elements $\lambda(x) \in H(\overline{S})$, and even with the induced elements of $H(T)$.

Define

$$y_1 * y_2 = y_1 \sigma_{x_{d_1}}(y_2)(x_{d_1} \star x_{d_2} \star x_{d_1 d_2}^{-1}).$$

We will verify that this definition makes $(H(T), *)$ into a group. The reader should be aware of the possible notational confusion resulting from the fact that there are three binary operations in use: $*$, $\star$, and juxtaposition. First we make some notational simplifications. Let $x_i = x_{d_i}$, $\sigma_i = \sigma_{x_i}$, $x_{ij} = x_{d_i d_j}$, $\sigma_{ij} = \sigma_{x_{ij}}$, and so forth. With this notation,

$$y_1 * y_2 = y_1 \sigma_1(y_2)(x_1 \star x_2 \star x_{12}^{-1}).$$

Note that $x_{12} = x_1 \star x_2 \star (x_2^{-1} \star x_1^{-1} \star x_{12})$ so

$$\sigma_{12} = \sigma_1 \circ \sigma_2 \circ \sigma_{x_2^{-1} \star x_1^{-1} \star x_{12}},$$

and the rightmost operand acts by conjugation, because of the axiom $\psi = \chi \circ i$ given in §3. Also note that if $x, x' \in g^{-1}(1)$ then $xx' = x \star x'$. Thus

$$(y_1 * y_2) * y_3$$

$$= y_1 \sigma_1(y_2)(x_1 \star x_2 \star x_{12}^{-1}) \sigma_1(\sigma_2((x_2^{-1} \star x_1^{-1} \star x_{12}) y_3(x_{12}^{-1} \star x_1 \star x_2)))$$
$$\quad (x_{12} \star x_3 \star x_{123}^{-1})$$

$$= y_1 \sigma_1(y_2)(x_1 \star x_2 \star x_{12}^{-1}) \sigma_1(\sigma_2(x_2^{-1} \star x_1^{-1} \star x_{12})) \sigma_1(\sigma_2(y_3))$$
$$\quad \sigma_1(\sigma_2(x_{12}^{-1} \star x_1 \star x_2))(x_{12} \star x_3 \star x_{123}^{-1})$$

$$= y_1 \sigma_1(y_2)(x_1 \star x_2 \star x_{12}^{-1})(x_{12} \star x_2^{-1} \star x_1^{-1}) \sigma_1(\sigma_2(y_3))$$
$$\quad (x_1 \star x_2 \star x_{12}^{-1} \star x_1 \star x_2 \star x_2^{-1} \star x_1^{-1})(x_{12} \star x_3 \star x_{123}^{-1})$$

$$= y_1 \sigma_1(y_2) \sigma_1(\sigma_2(y_3))(x_1 \star x_2 \star x_3 \star x_{123}^{-1})$$

$$= y_1 \sigma_1(y_2) \sigma_1(\sigma_2(y_3))(x_1 \star x_2 \star x_3 \star x_{23}^{-1} \star x_1^{-1})(x_1 \star x_{23} \star x_{123}^{-1})$$

$$= y_1 \sigma_1(y_2) \sigma_1(\sigma_2(y_3)) \sigma_1(x_2 \star x_3 \star x_{23}^{-1})(x_1 \star x_{23} \star x_{123}^{-1})$$

$$= y_1 \sigma_1[y_2 \sigma_2(y_3)(x_2 \star x_3 \star x_{23}^{-1})](x_1 \star x_{23} \star x_{123}^{-1})$$

$$= y_1 * [y_2 \sigma_2(y_3)(x_2 \star x_3 \star x_{23}^{-1})]$$

$$= y_1 * (y_2 * y_3),$$

which shows that $(H(T), *)$ is associative.

The identity element of $(H(T), *)$ is the identity element of $\overline{K}_1(T) = \overline{K}(T)$. We will now prove that inverses exist in $(H(T), *)$. It is useful to adopt the following barbarisms: $x_{1^{-1}}$ denotes $x_{d_1^{-1}}$ and $\sigma_{1^{-1}}$ denotes $\sigma_{x_{1^{-1}}}$. Define an element $\widetilde{y}_1 \in H(T)$ by first thinking of y_1 as an element of $\overline{K}(T)$, inverting it there, and then transporting the result to $\overline{K}_{d^{-1}}(T)$. Then $y_1 \widetilde{y}_1 = \widetilde{y}_1 y_1 = 1$. Let

$$y_1' = \sigma_1^{-1}(\widetilde{y}_1)(x_{1^{-1}}^{-1} \star x_1^{-1}).$$

Then

$$y_1 * y_1' = y_1 \sigma_1[\sigma_1^{-1}(\widetilde{y}_1)(x_{1^{-1}}^{-1} \star x_1^{-1})](x_1 \star x_{1^{-1}}) = \sigma_1(x_{1^{-1}}^{-1} \star x_1^{-1})(x_1 \star x_{1^{-1}})$$

$$\qquad = x_1 \star x_{1^{-1}}^{-1} \star x_1^{-1} \star x_1^{-1} \star x_1 \star x_{1^{-1}} \qquad = x_1 \star x_{1^{-1}}^{-1} \star x_1^{-1} \star x_{1^{-1}}$$

$$\qquad = x_1 \star x_1^{-1} \star x_{1^{-1}}^{-1} \star x_{1^{-1}} \qquad\qquad\qquad = 1,$$

and:

$$
\begin{aligned}
y_1' * y_1 &= \sigma_1^{-1}(\widetilde{y_1})(x_{1-1}^{-1} \star x_1^{-1})\sigma_{1-1}(y_1)(x_{1-1} \star x_1)\\
&= (x_{1-1}^{-1} \star x_1^{-1})(x_1 \star x_{1-1})\sigma_1^{-1}(\widetilde{y_1})(x_{1-1}^{-1} \star x_1^{-1})\sigma_{1-1}(y_1)(x_{1-1} \star x_1)\\
&= (x_{1-1}^{-1} \star x_1^{-1})\sigma_{x_1 \star x_{1-1}}(\sigma_{x_1^{-1}}(\widetilde{y_1}))\sigma_{1-1}(y_1)(x_{1-1} \star x_1)\\
&= (x_{1-1}^{-1} \star x_1^{-1})\sigma_{x_{1-1} \star x_1}(\sigma_{x_1^{-1}}(\widetilde{y_1}))\sigma_{1-1}(y_1)(x_{1-1} \star x_1)\\
&= (x_{1-1}^{-1} \star x_1^{-1})\sigma_{1-1}(\widetilde{y_1})\sigma_{1-1}(y_1)(x_{1-1} \star x_1)\\
&= x_{1-1}^{-1} \star x_1^{-1} \star x_{1-1} \star x_1 \;=\; 1,
\end{aligned}
$$

as above. Hence inverses exist in $(H(T), *)$. Thus $(H(T), *)$ is a group. Furthermore, the group structure is compatible with morphisms $T' \to T$ of $\overline{S}$-spaces. Hence this definition makes H into a group space.

We now show that λ defines a group isomorphism $\lambda\colon (X, \star) \to (H(\overline{S}), *)$. Suppose that $w_1 \in g^{-1}(d_1)$ and $w_2 \in g^{-1}(d_2)$. Then:

$$
\begin{aligned}
\lambda(w_1) * \lambda(w_2) &= [w_1 \star x_1^{-1}]_{d_1} * [w_2 \star x_2^{-1}]_{d_2}\\
&= [w_1 \star x_1^{-1}]_{d_1}\sigma_1([w_2 \star x_2^{-1}]_{d_2})(x_1 \star x_2 \star x_{12}^{-1})\\
&= [w_1 \star x_1^{-1}]_{d_1}[x_1 \star w_2 \star x_2^{-1} \star x_1^{-1}]_{d_2}(x_1 \star x_2 \star x_{12}^{-1})\\
&= [w_1 \star w_2 \star x_{12}^{-1}]_{d_{12}}\\
&= \lambda(w_1 \star w_2)
\end{aligned}
$$

so λ is a homomorphism. We leave it to the reader to verify that λ is an isomorphism. Taking account of our identifications, we see that the statement

$$
w_1 * w_2 = w_1 \star w_2
$$

is correct for any $w_1, w_2 \in X$.

We define an action of π on H. Let $\beta \in \pi$. Let $d \in D(\overline{S})$. Define $\tilde{\beta}(d)\colon \overline{K}_d \to \overline{K}_{\beta(d)}$ to be the composite:

$$
\overline{K}_d \xrightarrow{\text{can}} \overline{K} \xrightarrow{\tilde{\beta}} \overline{K} \xrightarrow{\beta(x_d) \star x_{\beta(d)}^{-1}} \overline{K} \xrightarrow{\text{can}} \overline{K}_{\beta(d)},
$$

where the third morphism from the left is right multiplication by $\beta(x_d) \star x_{\beta(d)}^{-1}$. Letting $d \in D(\overline{S})$ vary, we obtain an automorphism $\tilde{\beta}\colon H \to H$ of algebraic spaces. (Notice that the symbol $\tilde{\beta}$ also stands for an automorphism of $\overline{K}$.)

Observe that $\tilde{\beta}$ is compatible with the group structure on $\overline{K}$ in the sense that the diagram

$$
\begin{array}{ccc}
\overline{K} \times_{\overline{S}} \overline{K} & \xrightarrow{\;\mu\;} & \overline{K}\\
\Big\downarrow{\scriptstyle \tilde{\beta} \times \tilde{\beta}} & & \Big\downarrow{\scriptstyle \tilde{\beta}}\\
\overline{K} \times_{\overline{S}} \overline{K} & \xrightarrow{\;\mu\;} & \overline{K}
\end{array}
$$

commutes, where μ is the multiplication map. It follows that for any $x \in \overline{K}(\overline{S})$, the composites

$$
\overline{K} \xrightarrow{\;x\;} \overline{K} \xrightarrow{\;\tilde{\beta}\;} \overline{K}
$$

and

$$\overline{K} \xrightarrow{\tilde{\beta}} \overline{K} \xrightarrow{\beta(x)} \overline{K}$$

are equal. Using this, one sees that the composite

$$\overline{K} \xrightarrow{\beta'(x_d) \star x_{\beta'(d)}^{-1}} \overline{K} \xrightarrow{\tilde{\beta}} \overline{K} \xrightarrow{\beta(x_{\beta'(d)}) \star \beta(\beta'(x_d))^{-1}} \overline{K}$$

is equal to $\tilde{\beta}$. Hence the composite

$$\overline{K} \xrightarrow{\tilde{\beta}\tilde{\beta}'} \overline{K} \xrightarrow{\beta(\beta'(x_d)) \star x_{\beta(\beta'(d))}^{-1}} \overline{K}$$

is equal to the composite

$$\overline{K} \xrightarrow{\tilde{\beta}'} \overline{K} \xrightarrow{\beta'(x_d) \star x_{\beta'(d)}^{-1}} \overline{K} \xrightarrow{\tilde{\beta}} \overline{K} \xrightarrow{\beta(x_{\beta'(d)}) \star x_{\beta(\beta'(d))}^{-1}} \overline{K}.$$

It follows that $\widetilde{\beta\beta'} = \tilde{\beta}\tilde{\beta}'$ (as automorphisms of H), and so we have defined a group action of π on the algebraic space H. Then π acts on H/S but certainly not on $H/\overline{S}$. In fact, the diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\tilde{\beta}} & H \\
\downarrow & & \downarrow \\
\overline{S} & \xrightarrow{\beta} & \overline{S}
\end{array}
$$

commutes.

By a theorem of Artin [1, 7.1, 7.2]; [2, 6.3], it follows that there exists an algebraic space G over S such that if $\overline{G} = G \times_S \overline{S}$ then $\overline{G} \cong H$ as algebraic spaces over $\overline{S}$.

Now that we have constructed the algebraic space G, we would like to know that it can be given the structure of an S-group space in such a way that $\overline{G} \cong H$ as $\overline{S}$-group spaces. To prove this we must show that $\tilde{\beta}$ is compatible with the group structure on H, in the sense that the diagram

$$
\begin{array}{ccc}
H \times_{\overline{S}} H & \xrightarrow{\mu} & H \\
\downarrow{\scriptstyle \tilde{\beta} \times \tilde{\beta}} & & \downarrow{\scriptstyle \tilde{\beta}} \\
H \times_{\overline{S}} H & \xrightarrow{\mu} & H
\end{array}
$$

commutes, where μ is the multiplication map. This is equivalent to showing that the corresponding diagram

$$
\begin{array}{ccc}
\overline{K}_{d_1} \times \overline{K}_{d_2} & \longrightarrow & \overline{K}_{d_1 d_2} \\
\downarrow & & \downarrow \\
\overline{K}_{\beta(d_1)} \times \overline{K}_{\beta(d_2)} & \longrightarrow & \overline{K}_{\beta(d_1 d_2)}
\end{array}
$$

commutes. Unwinding the definitions, we see that this is equivalent to showing that the diagram

$$\overline{K} \times \overline{K} \xrightarrow{1 \times \sigma_1} \overline{K} \times \overline{K} \xrightarrow{\mu} \overline{K} \xrightarrow{x_1 \star x_2 \star x_{12}^{-1}} \overline{K}$$

$$\downarrow{\check{\beta} \times \check{\beta}} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\check{\beta}}$$

$$\overline{K} \times \overline{K} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{K}$$

$$\downarrow{[\beta(x_1) \star x_{\beta(d_1)}^{-1}] \times [\beta(x_2) \star x_{\beta(d_2)}^{-1}]} \qquad\qquad \downarrow{\beta(x_{12}) \star x_{\beta(d_1 d_2)}^{-1}}$$

$$\overline{K} \times \overline{K} \xrightarrow{1 \times \sigma_{x_{\beta(d_1)}}} \overline{K} \times \overline{K} \xrightarrow{\mu} \overline{K} \xrightarrow{x_{\beta(d_1)} \star x_{\beta(d_2)} \star x_{\beta(d_1 d_2)}^{-1}} \overline{K}$$

commutes. Calculating, one sees that this is equivalent to showing that the diagram

$$\overline{K} \times \overline{K} \xrightarrow{1 \times \sigma_1} \overline{K} \times \overline{K} \xrightarrow{\check{\beta} \times \check{\beta}} \overline{K} \times \overline{K}$$

$$\downarrow{\check{\beta} \times \check{\beta}} \qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\check{\mu}}$$

$$\overline{K} \times \overline{K} \qquad\qquad\qquad\qquad\qquad\qquad \overline{K}$$

$$\downarrow{[\beta(x_1) \star x_{\beta(d_1)}^{-1}] \times [\beta(x_2) \star x_{\beta(d_2)}^{-1}]} \qquad \downarrow{\beta(x_1) \star \beta(x_2) \star x_{\beta(d_2)}^{-1} \star x_{\beta(d_1)}^{-1}}$$

$$\overline{K} \times \overline{K} \xrightarrow{1 \times \sigma_{x_{\beta(d_1)}}} \overline{K} \times \overline{K} \xrightarrow{\mu} \overline{K}$$

commutes.

Since $\sigma: X \to \mathrm{Aut}(\overline{K})$ is a π-homomorphism, the composites

$$\overline{K} \xrightarrow{\sigma_1} \overline{K} \xrightarrow{\check{\beta}} \overline{K}$$

and

$$\overline{K} \xrightarrow{\check{\beta}} \overline{K} \xrightarrow{\sigma_{\beta(x_1)}} \overline{K}$$

are equal. In particular, the first row of the above diagram may be expressed as

$$\overline{K} \times \overline{K} \xrightarrow{\check{\beta} \times \check{\beta}} \overline{K} \times \overline{K} \xrightarrow{1 \times \sigma_{\beta(x_1)}} \overline{K} \times \overline{K}.$$

Thus we may reduce to showing that the diagram:

$$\overline{K} \times \overline{K} \xrightarrow{\qquad 1 \times \sigma_{\beta(x_1)} \qquad} \overline{K} \times \overline{K}$$

$$\downarrow{[\beta(x_1) \star x_{\beta(d_1)}^{-1}] \times [\beta(x_2) \star x_{\beta(d_2)}^{-1}]} \qquad\qquad \downarrow{\mu}$$

$$\overline{K} \times \overline{K} \qquad\qquad\qquad\qquad\qquad \overline{K}$$

$$\downarrow{1 \times \sigma_{x_{\beta(d_1)}}} \qquad\qquad\qquad\qquad \downarrow{\beta(x_1) \star \beta(x_2) \star x_{\beta(d_2)}^{-1} \star x_{\beta(d_1)}^{-1}}$$

$$\overline{K} \times \overline{K} \xrightarrow{\qquad \mu \qquad} \overline{K}$$

commutes. Fortunately, all morphisms in this diagram are $\overline{S}$-morphisms. Hence to prove that it commutes, it suffices to prove that it commutes on all

T-valued points. Indeed, let y, $y' \in \overline{K}(T)$. We must show that

$$y[\beta(x_1) \star x_{\beta(d_1)}^{-1}]\sigma_{x_{\beta(d_1)}}(y'[\beta(x_2) \star x_{\beta(d_2)}^{-1}])$$
$$= y\sigma_{\beta(x_1)}(y')[\beta(x_1) \star \beta(x_2) \star x_{\beta(d_2)}^{-1} \star x_{\beta(d_1)}^{-1}].$$

This is equivalent to showing that

$$[\beta(x_1) \star x_{\beta(d_1)}^{-1}]\sigma_{x_{\beta(d_1)}}(y')[x_{\beta(d_1)} \star \beta(x_2) \star x_{\beta(d_2)}^{-1} \star x_{\beta(d_1)}^{-1}])$$
$$= \sigma_{\beta(x_1)}(y')[\beta(x_1) \star \beta(x_2) \star x_{\beta(d_2)}^{-1} \star x_{\beta(d_1)}^{-1}].$$

This follows easily once we know that

$$\sigma_x(y) = x * y * x^{-1}$$

for any $x \in X$ and any $y \in H(T)$. We proceed to establish this. We may assume that $y \in \overline{K}_1(T)$. Let $d_1 = g(x)$. The special case where $d_1 = 1$ is true, because of the axiom $\psi = \chi \circ i$ given in §3. The special case where $x = x_1$ is also true because

$$x_1 * y * x_1^{-1} = [x_1\sigma_1(y)] * x_1^{-1} = x_1\sigma_1(y)\sigma_1(x_1^{-1})(x_1 \star x_{1^{-1}})$$
$$= x_1\sigma_1(y)x_1^{-1}(x_1 \star x_{1^{-1}})$$
$$= x_1\sigma_1(y)(x_1^{-1} \star x_{1^{-1}}^{-1}) \star (x_1 \star x_{1^{-1}})x_{1^{-1}}$$
$$= x_1\sigma_1(y)x_{1^{-1}} = \sigma_1(y).$$

In the general case,

$$\sigma_x(y) = \sigma_{x*x_1^{-1}*x_1}(y) = \sigma_{x*x_1^{-1}}[\sigma_1(y)] = \sigma_{x*x_1^{-1}}(x_1 * y * x_1^{-1})$$
$$= (x \star x_1^{-1}) * (x_1 * y * x_1^{-1}) * (x_1 \star x^{-1}) = x * y * x^{-1}.$$

Hence $\tilde{\beta}$ is compatible with group structure on H, in the sense described above. Hence H descends to a group space G over S.

In fact we have an extension

$$\xi: 1 \to \overline{K} \to H \to \overline{D} \to 1$$

of group spaces over $\overline{S}$. As π acts on everything in sight, we obtain an $\overline{S}$-decomposed extension

$$\xi_0: 1 \to K \to G \to D \to 1$$

of group spaces over S.

We leave it to the reader to verify that this construction is in fact inverse to the first construction. □

5. DECOMPOSITION BY FINITE ÉTALE MORPHISMS

We consider three cases under which every extension G of D by K is isodecomposable, where K has geometrically connected fibers, and D is an isodecomposable étale covering space. In the third case, we consider only commutative extensions.

The reader will observe that this extension problem is equivalent to determining when certain principal homogeneous spaces (namely the cosets of the extension) are decomposed by some finite étale surjective morphism. However,

for fixed D and K, we do not know exactly which principal homogeneous spaces can occur as the cosets of some extension of D by K.

Case I: S is the spectrum of a perfect field (obvious)

In this case we note that for suitable K and D, there may not exist a single finite étale surjective morphism which decomposes every extension of D by K:

Example 5.1. Let k be a number field and let E be an elliptic curve over k such that $E(k)$ contains an element of order 2. By the Mordell-Weil theorem, $E(k)$ is finitely generated. Under these hypotheses, [13, Theorem 7], shows that the group of principal homogeneous spaces under E contains infinitely many elements of order 2. On the other hand, if K is a finite field extension of k, then since $E(K)$ is finitely generated, the group $H^1(K/k, E)$ of K-split principal homogeneous spaces under E is finite. Hence no single field extension will decompose every order 2 principal homogeneous space under E. For any such principal homogeneous space X, we can use (2.1) to construct an extension G of $\mathbb{Z}/2\mathbb{Z}$ by E such that $G = E \cup X$, scheme-theoretically. Indeed, one has an exact sequence

$$E(\overline{k}) \to Z^1(S, E) \to \mathrm{PHS}(E) \to 0$$

where $\overline{k}$ is an algebraic closure of k and $S = \mathrm{Spec}\, k$. Since $E(\overline{k})$ is divisible, $[X] \in \mathrm{PHS}(E)$ lifts to a 2-torsion element of $Z^1(S, E)$. This allows one to construct the extension G. Therefore, no single field extension of k will decompose every extension of $\mathbb{Z}/2\mathbb{Z}$ by E.

Case II: S is the spectrum of an Artin local ring and K is smooth.

In this case we prove the following stronger statement:

Lemma 5.2. *Let S be the spectrum of an Artin local ring. Let K be a smooth group space over S. Let H be a principal homogeneous space under K. Then H is isotrivial, by which we mean that H becomes trivial after base extension by a finite étale surjective morphism $\overline{S} \to S$.*

This follows immediately from the following:

Lemma 5.3. *Let S be the spectrum of an Artin local ring. Let $f: X \to S$ be a smooth morphism, $X \neq \varnothing$. Then there exists a finite surjective étale morphism $S' \to S$ such that $f \times_S S'$ admits a section.*

Proof. The induced map $X_{\mathrm{red}} \to S_{\mathrm{red}}$ is smooth. Write $S_{\mathrm{red}} = \mathrm{Spec}\, k$. By [10, 17.15.10(iii)], there exists some $x \in X_{\mathrm{red}}$ such that $k' = k(x)$ is a finite separable extension of k. Let $S'_{\mathrm{red}} = \mathrm{Spec}\,(k')$. Then the induced map $X'_{\mathrm{red}} \to S'_{\mathrm{red}}$ admits a section. By [7, IX, 1.7], there exists a finite étale morphism $S'' \to S$ such that $S'_{\mathrm{red}} = S_{\mathrm{red}} \times_S S'$. By the definition of formal smoothness, $f \times_S S'$ admits a section. $\square$

Remark 5.4. Case II is false without the hypothesis that K is smooth. Indeed, let k be an algebraically closed field of characteristic $p > 0$. Let $A = k[x]/(x^p)$. Let $S = \mathrm{Spec}(A)$. Let $K = \alpha_p$. Then $\mathrm{PHS}(K) = A/A^p$. (See e.g. [16, p. 128].) In particular, there exists a nontrivial principal homogeneous space X under K. By (2.1), one may construct an extension of group spaces:

$$0 \to K \to G \xrightarrow{\pi} D \to 0$$

where D is a cyclic group (with the discrete scheme structure) and $\pi^{-1}(d) \cong X$, where d is a generator for D. Every étale cover of S is trivial, so G is not isodecomposable.

Definition. Let G be a smooth commutative group space over S, having geometrically connected fibers. Then G is *nondegenerate* if the functions $s \mapsto \dim(G_s)_{\mathrm{ab}}$, $s \mapsto \dim(G_s)_{\mathrm{mul}}$, and $s \mapsto \dim(G_s)_{\mathrm{unip}}$ are constant. (Here s denotes a geometric point of S and the subscripts ab, mul, and unip refer to the abelian, multiplicative, and unipotent parts of G_s.)

Remark 5.5. If S is a complex variety, and G is smooth, commutative, and has geometrically connected fibers, then G is nondegenerate if and only if its fibers are all diffeomorphic.

Example 5.6. Any abelian scheme is nondegenerate.

Case III: D is finite, $|D|$ is relatively prime to the residue characteristics of S, K is smooth, commutative, separated, and nondegenerate; assuming that G is commutative

The following statement generalizes the corresponding statement for abelian schemes, which is well known (see e.g. [17, 20.7]).

Proposition 5.7. *Let G be a smooth, commutative group space over S, separated, having connected fibers, and nondegenerate. Let n be a positive integer, relatively prime to the residue characteristics of S. Let $Q = \mathrm{Ker}(G \overset{n}{\to} G)$. Then the canonical map $Q \to S$ is finite, étale, and surjective.*

Proof. Since a flat morphism with étale fibers is étale, it suffices by [10, 18.2.9] to show that Q/S is flat and that its geometric fibers are étale, of some fixed finite rank.

In this paragraph we suppose that S is the spectrum of a field k. I claim that $n: G \to G$ is flat and has an étale kernel. For this we may suppose that k is algebraically closed. By generic flatness, n is flat at at least one point. By [4, 1.3], n is flat. Now we show that n is surjective. Given an exact sequence

$$(*) \qquad\qquad 0 \to G' \to G \to G'' \to 0$$

for which n' and n'' are surjective, it will follow that n is surjective. By the structure theory of smooth commutative group schemes over an algebraically closed field, we may reduce to $G \in \{\mathbb{G}_a, \mathbb{G}_m, A\}$, where A is an abelian variety. The cases $G \in \{\mathbb{G}_a, \mathbb{G}_m\}$ are obvious. The case where $G = A$ is well known: see e.g. [17, 8.2]. Hence n is surjective for any G. It follows that given an exact sequence $(*)$ as above, one obtains an exact sequence

$$0 \to Q' \to Q \to Q'' \to 0$$

of kernels. It follows by [4, 9.2(viii)] that if Q' and Q'' are étale, then so is Q. Thus to prove Q is étale, we may reduce to $G \in \{\mathbb{G}_a, \mathbb{G}_m, A\}$. The cases $G \in \{\mathbb{G}_a, \mathbb{G}_m\}$ are obvious. The case where $G = A$ is well known: see e.g. [17, 8.2]. Hence Q is étale for any G.

Suppose that S is arbitrary. By the preceding paragraph, n is flat on every fiber. Since G is flat over S, it follows by [7, IV, 5.9] that n is a flat morphism. Hence Q is flat over S.

It remains to show that the geometric fibers of Q/S do not vary in their rank. This is left as an exercise for the reader: use the fact that G/S is nondegenerate. □

Corollary 5.8. *Let K be a smooth, commutative group space over S, separated, having connected fibers, and nondegenerate. Let D be a finite étale group scheme over S. Assume that the rank of D is relatively prime to the residue characteristics of S. Let*

$$0 \to K \to G \to D \to 0$$

be an extension of commutative group schemes. Then G is isodecomposible.

Proof. We may reduce to the case where D is already decomposable. Let $d \in D(S)$. Let $\pi: G \to D$ be the canonical map. Let $G_d = \pi^{-1}(d)$. It suffices to construct a finite étale surjective morphism $S' \to S$ such that $G'_d \to S'$ admits a section. Let n be the rank of D. Let M_d be such that the following diagram is cartesian:

$$
\begin{array}{ccc}
M_d & \longrightarrow & G_d \\
\downarrow{\scriptstyle \phi_d} & & \downarrow{\scriptstyle n} \\
S & \xrightarrow{\;0\;} & K.
\end{array}
$$

If G_d admits a section then (5.7) implies that ϕ_d is a finite étale surjective morphism, since in that case M_d is isomorphic to Q as an S-space. Even if G_d does not admit a section, we can conclude by faithfully flat descent that ϕ_d is a finite étale surjective morphism. Let $S' = M_d$. Then $M'_d = M_d \times_S S'$ admits a section over S'. Hence G'_d admits a section. □

6. Examples

We consider a few examples where $S = \mathrm{Spec}(\mathbb{Z})$. In that case every étale covering group over S is decomposable, e.g. by [14, p. 137] and (1.1).

Proposition 6.1. *Let $S = \mathrm{Spec}(\mathbb{Z})$. Let K be one of the following group schemes over S: either $\mathbb{G}_a^n$ for some $n \geq 1$, or SL_n for some odd integer $n \geq 3$. Let D be any étale covering group over S. Then group scheme extensions of D by K (up to equivalence) are in bijective correspondence with group extensions of $D(S)$ by $K(S)$ (up to equivalence).*

Proof. By (4.1), it suffices to show that (a) every principal homogeneous space under K is trivial, and that (b) the induced map $\psi: \mathrm{Aut}(K) \to \mathrm{Aut}(K(S))$ is an isomorphism. First we consider (a). In case $K = \mathbb{G}_a$, it is well known that $\mathrm{PHS}(K) = H^1(S, \mathscr{O}_S)$, which is zero for $S = \mathrm{Spec}(\mathbb{Z})$, so every principal homogeneous space under K is trivial. The same statement holds for $K = \mathbb{G}_a^n$. Since every vector bundle over S is trivial, $\mathrm{PHS}(\mathrm{GL}_n) = 0$. From the six term exact sequence of pointed sets [16, III, 4.5] associated to the exact sequence

$$1 \to \mathrm{SL}_n \to \mathrm{GL}_n \to \mathbb{G}_m \to 1,$$

one concludes that $\mathrm{PHS}(\mathrm{SL}_n) = 0$.

Now we consider (b). In case $K = \mathbb{G}_a^n$, this is clear. Suppose $K = \mathrm{SL}_n$, for some odd integer $n \geq 3$. By [11, Theorem 3, Remark], one knows that $\mathrm{Aut}(\mathrm{SL}_n(\mathbb{Z}))$ is generated by conjugation automorphisms and by the automorphism $M \mapsto (M^t)^{-1}$. Thus it is clear that ψ is surjective. Let $\sigma \in \mathrm{Aut}(\mathrm{SL}_n)$,

$\sigma \neq id$. Then the induced automorphism $\sigma \otimes \mathbb{Q} \in \text{Aut}(\text{SL}_n \otimes \mathbb{Q})$ is not the identity. One knows that $\text{SL}_n(\mathbb{Z})$ is Zariski-dense in $\text{SL}_n(\mathbb{Q})$. (This follows e.g. from the fact that $\text{SL}_n(\mathbb{Q})$ is generated by elementary matrices — see e.g. [3, V, 9.2], but this fact is completely elementary.) Hence there is some $M \in \text{SL}_n(\mathbb{Z})$ on which $\sigma \otimes \mathbb{Q}$ acts nontrivially. Hence $\psi(\sigma)$ is not the identity. Hence ψ is injective. $\square$

Proposition 6.2. *Let* $S = \text{Spec}(\mathbb{Z})$. *Let* D *be any étale covering group over* S. *Then group scheme extensions of* D *by* $\mathbb{G}_m$ *(up to equivalence) are in bijective correspondence with pairs* (η, f) *consisting of a group extension* η *of* $D(S)$ *by* $\mathbb{Z}/2\mathbb{Z}$ *(up to equivalence) and a group homomorphism* $f: D \to \mathbb{Z}/2\mathbb{Z}$.

Proof. Since $\text{Pic}(\mathbb{Z}) = 0$, $\text{PHS}(\mathbb{G}_m) = 0$. Also $\text{Aut}(\mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$, which acts trivially on $\mathbb{G}_m(\mathbb{Z}) = \{1, -1\} = \mathbb{Z}/2\mathbb{Z}$. The conclusion follows from (4.1). $\square$

7. Obstructions to isodecomposability

Let G be a group space over S. We consider various reasons why G/S might not be isodecomposable.

(I) It may not be possible to find an exact sequence of group spaces

$$1 \to K \to G \to D \to 1$$

such that K has geometrically connected fibers and D/S is étale. One can always define a subfunctor G^0 of G (as in [4, §3], but it will not always be representable by an algebraic space K. It is representable if and only if one can find an exact sequence as above. This holds (for instance) if G/S is smooth. (See [4, 3.10].) The simplest example where G^0 is not representable is obtained by taking $S = \text{Spec}(\mathbb{Z})$, $G = \mu_p = \text{Spec}(\mathbb{Z}[x]/(x^p - 1))$, for any prime p. Another example is obtained by taking $S = \text{Spec}(A)$, where A is a discrete valuation ring of equal characteristic $p > 0$, with uniformizing parameter t, and G to be the kernel of the homomorphism $\mathbb{G}_a \to \mathbb{G}_a$ given by $x \mapsto x^p - tx$. (See [6, #236, §1.9].) In both of these examples G is finite and flat.

(II) Even if it is possible to find an exact sequence of group spaces as in (I), it may be that D is not locally decomposable. We may take $G = D$. For example, if G'/S is a finite étale group scheme, and $G \subset G'$ is an open subgroup-scheme, then G will almost never be locally decomposable. Also G/G' will be usually be nonseparated, and hence also not locally decomposable. For a specific example of this, take $G = D$, $S = \mathbb{A}^1 = \text{Spec}(\mathbb{C}[x])$, G' to be the constant S-group $\mathbb{Z}_2$, and $G \subset G'$ to be the open subgroup scheme obtained by deleting the nonzero point from the fiber over $0 \in \mathbb{A}^1$.

(III) Even if the problems of (I) and (II) do not occur, it may be that D cannot be decomposed by a finite étale morphism. (See 1.7.) Probably this can only happen when the fibers of D are not finitely generated as groups.

(IV) Even if the exact sequence exists, and D is decomposed by a finite étale morphism, it may be that G is not decomposed by a finite étale morphism. To consider this possibility, we may suppose that D is itself decomposable. Then the cosets of K in G are principal homogeneous spaces under K. By (2.1), every principal homogeneous space occurs as a coset of some extension. Hence it suffices to give examples of principal homogeneous spaces which are not isotrivial. We give three examples:

(a) If k is any imperfect field, there is an example in which $S = \mathrm{Spec}\, k$ and K is finite. One may take $K = \alpha_p$. (Cf. (5.4).)

(b) If k is an algebraically closed field of positive characteristic, and $S = \mathrm{Spec}\, k[x]/(x^p)$, there is an example (5.4) in which K is finite.

(c) Let $K = \mathbb{G}_m$. Let $D = \mathbb{Z}$. For a suitable choice of S, there will be a line bundle $\mathscr{L}$ of infinite order on S. Since $\mathrm{PHS}(\mathbb{G}_m) \cong \mathrm{Pic}(S)$, one can use (2.2) to build an exact sequence

$$0 \to \mathbb{G}_m \to G \to \mathbb{Z} \to 0$$

of commutative group schemes. Now in order that $p \colon \overline{S} \to S$ decomposes this, it is necessary and sufficient that $p^*\mathscr{L} \cong \mathscr{O}_{\overline{S}}$. This will never happen if p is finite étale surjective.

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